Absolute Convergence of Rational Series is Semi-decidable

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Abstract. We study *real-valued absolutely convergent rational series*, i.e. functions $r : \Sigma^* \to \mathbb{R}$, defined over a free monoid Σ^*, that can be computed by a multiplicity automaton A and such that $\sum_{w \in \Sigma^*} |r(w)| < \infty$. We prove that any absolutely convergent rational series r can be computed by a multiplicity automaton A which has the property that $r_{|A|}$ is simply convergent, where $r_{|A|}$ is the series computed by the automaton $|A|$ derived from A by taking the absolute values of all its parameters. Then, we prove that the set $\mathcal{A}^{rat}(\Sigma)$ composed of all absolutely convergent rational series is semi-decidable and we show that the sum $\sum_{w \in \Sigma^*} |r(w)|$ can be estimated to any accuracy rate for any $r \in \mathcal{A}^{rat}(\Sigma)$. We also introduce a spectral radius-like parameter $\rho_{|r|}$ which satisfies the following property: r is absolutely convergent iff $\rho_{|r|} < 1$.

1 Introduction

Given a finite alphabet Σ, we consider real formal power series defined over the free monoid Σ^*, i.e. functions which map Σ^* into $\mathbb{R}$. More precisely, we consider *rational series*, which admit several characterizations, one of which being that they can be computed by multiplicity automata [1, 2]. Given a rational series $r : \Sigma^* \to \mathbb{R}$, we study whether r is absolutely convergent, i.e. $\sum_{w \in \Sigma^*} |r(w)| < \infty$. It is polynomially decidable whether a rational series r is simply convergent, i.e. whether the sum $\sum_{n \geq 0} \sum_{w \in \Sigma^n} r(w)$ converges to a limit (see [3] for example). Since the Hadamard product r of two rational series s and t, defined by $r(w) = s(w)t(w)$, is rational, it is polynomially decidable whether a rational series r converges in quadratic norm, i.e. $\sum_{w \in \Sigma^*} r^2(w) < \infty$. However, to our knowledge, it is still unknown whether it can be decided if a rational series is absolutely convergent.

The motivation for the present work comes from a problem we have studied in grammatical inference. A *stochastic language* over Σ^* is a series p which takes only non negative values and s.t. $\sum_{w \in \Sigma^*} p(w) = 1$[1]. A classical problem in grammatical inference consists in inferring an estimate of a target stochastic language p from a finite sample of words $\{w_1, \ldots, w_n\}$ independently drawn

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[1] This definition differs from the one given in [2]

according to p. In [4], we proposed an algorithm DEES which takes a sample as input and outputs a rational series r which simply converges to 1 but can take negative values, and which satisfies the following property: with probability one, there exists a sample size level from which the sum $\sum_{w\in\Sigma^*}|p(w)-r(w)|$ is arbitrarily small, (which implies that r is absolutely convergent); moreover, a stochastic language p_r can be computed from r, that satisfies

$$\sum_{w\in\Sigma^*}|p_r(w)-r(w)|\leq\sum_{w\in\Sigma^*}|r|(w)-1.$$

In other words, we know that from some sample size, we will have a solution of our problem. But we need to decide whether the series r output by DEES from the working sample is absolutely convergent to ensure that r provides a solution and we need to compute an estimate of $\sum_{w\in\Sigma^*}|r|(w)$ to bound the accuracy of this solution.

A *multiplicity automaton (MA)* is a tuple $A=\langle\Sigma,Q,\varphi,\iota,\tau\rangle$, where Q is a finite set of states, φ (resp. ι, τ) is a transition (resp. initialization, termination) function, which can be used to compute a rational series r_A. Any rational series r can be computed by a MA. Given a MA A, we obtain a new MA $|A|$ by taking the absolute values of the functions φ,ι and τ. It is straightforward that r_A is absolutely convergent if $r_{|A|}$ is simply convergent. We prove that any absolutely convergent rational series r can be computed by a multiplicity automaton A which has the property that $r_{|A|}$ is simply convergent. Then, we provide an algorithm which takes a multiplicity automaton B as input and halts if and only if r_B is absolutely convergent: when the algorithm halts, it outputs a MA A equivalent to B, i.e. the series computed by A and B are equal, and such that $r_{|A|}$ is convergent. So, we have proved that the set $\mathcal{A}^{rat}(\Sigma)$ composed of all absolutely convergent rational series is semi-decidable.

The sum $\sum_{w\in\Sigma^*}|r(w)|$ can be estimated from below by computing the sum $\sum_{w\in\Sigma^{\leq n}}|r(w)|$ for increasing integers n. We prove that our algorithm can be used to provide convergent upper bounds. So, the sum $\sum_{w\in\Sigma^*}|r(w)|$ can be estimated to any accuracy rate. As a consequence, if the $L1$-distance $||r-s||_1=\sum_{w\in\Sigma^*}|r(w)-s(w)|$ is finite, it can be estimated to any accuracy rate. It has been proved in [5] that computing the $L1$-distance between two hidden Markov models is NP-hard which implies that computing the $L1$-distance between two MA is NP-hard too. L_p distances of two probabilistic automata have been studied in [6, 7], where efficient algorithm have been provided when p *is even*. Our algorithm can be used to estimate L_p distances between two rational series for any odd values of p.

Finally, for any rational series r, we introduce a spectral radius-like parameter, $\rho_{|r|}$ defined by $\rho_{|r|}=limsup_n(|r|(\Sigma^n))^{1/n}$ and we show that r is absolutely convergent iff $\rho_{|r|}<1$.

We recall some properties on rational series and multiplicity automata in Section 2. We study absolutely convergent series in Section 3; in particular, we prove that any absolutely convergent rational series can be represented by a MA A such that $r_{|A|}$ is convergent. We prove the semi-decidability of the class

$\mathcal{A}^{rat}(\Sigma)$ in Section 4. We show how the sum $\sum_{w\in\Sigma^*}|r(w)|$ can be estimated in Section 5. To conclude, we provide some comments and describe some conjectures and future works in Section 6.

2 Preliminaries

2.1 Rational Series

Let Σ^* be the set of words on the finite alphabet Σ. The empty word is denoted by ε, and the length of a word u is denoted by $|u|$. For any integer k, we denote by Σ^k the set $\{u \in \Sigma^* \mid |u| = k\}$ and by $\Sigma^{\leq k}$ the set $\{u \in \Sigma^* \mid |u| \leq k\}$. A subset S of Σ^* is *prefix-closed* if for any $u, v \in \Sigma^*$ $uv \in S \Rightarrow u \in S$.

The general context is, for an alphabet Σ the set $\mathbb{R}\langle\langle\Sigma\rangle\rangle$ of all the mappings from Σ^* into $\mathbb{R}$. An element of this set is called a *formal power series*. This set is an $\mathbb{R}$-vector space. For any series r and any word $u \in \Sigma^*$, we denote by $\dot{u}r$ the series defined by $\dot{u}r(w) = r(uw)$. Let $LH(r)$ denote the *linear hull* of r, i.e. the vector subspace of $\mathbb{R}\langle\langle\Sigma\rangle\rangle$ spanned by $\{\dot{u}r|u \in \Sigma^*\}$.

A *multiplicity automaton (MA)* is a tuple $\langle\Sigma, Q, \varphi, \iota, \tau\rangle$ where Q is a finite set of states, $\varphi : Q \times \Sigma \times Q \to \mathbb{R}$ is the *transition function*, $\iota : Q \to \mathbb{R}$ is the *initialization function* and $\tau : Q \to \mathbb{R}$ is the *termination function*. Let $Q_I = \{q \in Q|\iota(q) \neq 0\}$ be the set of *initial states* and $Q_T = \{q \in Q|\tau(q) \neq 0\}$ be the set of *terminal states*. We extend the transition function φ to $Q \times \Sigma^* \times Q$ by $\varphi(q, wx, q') = \sum_{q''\in Q} \varphi(q, w, q'')\varphi(q'', x, q')$ and $\varphi(q, \varepsilon, q') = 1$ if $q = q'$ and 0 otherwise, for any $q, q' \in Q$, $x \in \Sigma$ and $w \in \Sigma^*$. For any finite subset $L \subset \Sigma^*$ and any $Q' \subseteq Q$, define $\varphi(q, L, Q') = \sum_{w\in L, q'\in Q'} \varphi(q, w, q')$. For any MA $A = \langle\Sigma, Q, \varphi, \iota, \tau\rangle$, we define the series r_A by $r_A(w) = \sum_{q,q'\in Q} \iota(q)\varphi(q, w, q')\tau(q')$. For any $q \in Q$, we define the series $r_{A,q}$ by $r_{A,q}(w) = \sum_{q'\in Q} \varphi(q, w, q')\tau(q')$. The *support* of a MA $\langle\Sigma, Q, \varphi, \iota, \tau\rangle$ is a non deterministic finite automaton (NFA) $\langle\Sigma, Q, \delta, Q_I, Q_F\rangle$ where the transition function is defined by $\delta(q, x) = \{q' \in Q|\varphi(q, x, q') \neq 0\}$.

We will say that a series r is *rational* if it satisfies one of the two following equivalent conditions:

1. the dimension of $LH(r)$ is finite;
2. r can be computed by a multiplicity automaton.

The family of all rational series is denoted by $\mathbb{R}^{rat}\langle\langle\Sigma\rangle\rangle$.

2.2 Prefixial Multiplicity Automata

Representation of rational series based on prefix sets has been introduced in [1].

Definition 1. *Let $A = \langle\Sigma, Q, \varphi, \iota, \tau\rangle$ be a MA. We say that A is* prefixial *if:*

- *Q is non-empty prefix-closed finite subset of Σ^**
- *$\forall u \in Q, \iota(u) \neq 0$ iff $u = \varepsilon$*
- *$\forall x \in \Sigma, \forall u, v \in Q$ s.t. $ux \in Q$, $\varphi(u, x, v) \neq 0$ iff $v = ux$.*

A transition (u, x, v) *is called an* inner transition *if* $v = ux$ *and a* border transition *otherwise. The set* $F = \{ux | u \in Q, x \in \Sigma, ux \notin Q\}$ *is called the* frontier set *of* A.

An NFA $A = \langle \Sigma, Q, \delta, I, F \rangle$ *is* prefixial *if* A *is the support of a prefixial MA.*

A prefix-closed subset Q of Σ^* can be used as the set of states of a prefixial automaton that computes a rational series r if and only if the set $\{\dot{u}r | u \in Q\}$ spans $LH(r)$. Let us make this statement precise.

Let r be a rational series and let Q be a prefix-closed subset such that $\{\dot{u}r | u \in Q\}$ spans $LH(r)$. Let $F = \{ux | u \in Q, x \in \Sigma, ux \notin Q\}$. Let $f : Q \to \mathbb{R}$ and $g : F \times Q \to \mathbb{R}$ be two mappings that satisfy:

1. $f(u) \neq 0$ for every state u,
2. $\dot{\overline{ux}}r = \sum_{v \in Q} g(ux, v) \frac{\overline{f}(u)}{\overline{f}(v)} \dot{v}r$ for every $ux \in F$ where $\overline{f}$ is defined by $\overline{f}(\varepsilon) = f(\varepsilon)$, where $\overline{f}(wx) = \overline{f}(w) f(wx)$ for any $w \in \Sigma^*$ and $x \in \Sigma$ and where the top bar notation is meant to express that the "dot" applies to the element under the bar. The function g expresses linear dependencies.

We define the prefixial automaton $A(\Sigma, Q, f, g, r) = \langle \Sigma, Q, \varphi, \iota, \tau \rangle$ by

- $\iota(\varepsilon) = f(\varepsilon)$,
- $\forall u \in Q, x \in \Sigma$ s.t. $ux \in Q$, $\varphi(u, x, ux) = f(ux)$,
- $\forall u, v \in Q, x \in \Sigma$ s.t. $ux \in F$, $\varphi(u, x, v) = g(ux, v)$,
- $\forall u \in Q, \tau(u) = \frac{r(u)}{\overline{f}(u)}$.

Proposition 1. *The automaton* $A(\Sigma, Q, f, g, r)$ *computes* r.

Proof. Let $A(\Sigma, Q, f, g, r) = \langle \Sigma, Q, \varphi, \iota, \tau \rangle$. Let us show, by induction on $|w|$, that for any $u \in Q$ and any $w \in \Sigma^*$, $\dot{u}r(w) = \overline{f}(u) r_{A,u}(w)$.

- Let $u \in Q$: $\dot{u}r(\varepsilon) = r(u)$ and $\overline{f}(u) r_{A,u}(\varepsilon) = \overline{f}(u)\tau(u) = r(u)$.
- Let $u \in Q$, $w \in \Sigma^*$ and $x \in \Sigma$.

If $ux \in Q$,
$$\begin{aligned} \overline{f}(u) r_{A,u}(xw) &= \overline{f}(u) \varphi(u, x, ux) r_{A,ux}(w) \\ &= \overline{f}(ux) r_{A,ux}(w) \text{ by definition of } \overline{f} \\ &= \overline{f}(ux) \frac{\dot{\overline{ux}}r(w)}{\overline{f}(ux)} \text{ by induction hypothesis} \\ &= \dot{\overline{ux}}r(w). \end{aligned}$$

If $ux \notin Q$,
$$\begin{aligned} \overline{f}(u) r_{A,u}(xw) &= \overline{f}(u) \sum_{v \in Q} g(ux, v) r_{A,v}(w) \text{ by definition of } A \\ &= \overline{f}(u) \sum_{v \in Q} g(ux, v) \frac{\dot{v}r(w)}{\overline{f}(v)} \text{ by induction hypothesis} \\ &= \dot{\overline{ux}}r(w) \text{ by definition of } g. \end{aligned}$$

Therefore, for any $u \in Q$, $\dot{u}r = \overline{f}(u) r_{A,u}$ and in particular, $r = f(\varepsilon) r_{A,\varepsilon} = r_A$.
□

3 On Representation of Absolutely Convergent Rational Series

3.1 Absolutely Convergent Rational Series

Let r be a series and let Γ be a non-empty subset of Σ^*. We say that r is *convergent* on Γ if the sum $\sum_{n\geq 0}\sum_{w\in\Gamma\cap\Sigma^n} r(w)$ is convergent; if so, we denote the sum by $r(\Gamma)$. Let $|r|$ be the series defined by $|r|(w) = |r(w)|$. We say that r is *absolutely convergent* if $|r|$ is convergent. Note that when a series r is absolutely convergent, it is convergent over any $\Gamma \subseteq \Sigma^*$.

We denote by $\mathcal{A}(\Sigma)$ (resp. by $\mathcal{A}^{rat}(\Sigma)$) the subspace of $\mathbb{R}\langle\langle\Sigma\rangle\rangle$ (resp. of $\mathbb{R}^{rat}\langle\langle\Sigma\rangle\rangle$) composed of the series that are absolutely convergent.

Let r be a series, we denote by $res(r)$ the following subset of Σ^*: $res(r) = \{u \in \Sigma^*/\exists w \in \Sigma^*, r(uw) \neq 0\}$. For any absolutely convergent series r and any word $u \in res(r)$, we denote by $u^{-1}r$ the series defined by $u^{-1}r(w) = \frac{|r|(\Sigma^*)}{|r|(u\Sigma^*)}\dot{u}r$ and we call it the *residual* of r associated with u. The set of all the residuals of r is denoted by $Res(r)$. The vector subspace spanned by $Res(r)$ is equal to $LH(r)$. Note that for any $u \in res(r), |u^{-1}r|(\Sigma^*) = |r|(\Sigma^*)$. The mapping $r \to |r|(\Sigma^*)$ defines a norm $||\cdot||_1$ on $\mathcal{A}(\Sigma)$. Let us denote by $CH(r)$ the convex hull of $Res(r)$: $CH(r) = \{\sum_{i=1}^n \alpha_i u_i^{-1}r | n \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1, u_i \in res(r)\}$. Let us denote by $CCH(r)$, the closed convex hull of $Res(r)$, i.e. the closure of $CH(r)$. Note that when $r \in \mathcal{A}^{rat}(\Sigma)$, $||\cdot||_1$ is constant on $CCH(r)$. In particular, $CCH(r)$ is a compact convex set.

Lemma 1. *Let $r \in \mathcal{A}^{rat}(\Sigma)$. Then $\forall\epsilon > 0, \exists k \in \mathbb{N}$ such that $\forall s \in CCH(r)$, $|s|(\Sigma^{>k}) < \epsilon$. In particular, $\forall\epsilon > 0, \exists k \in \mathbb{N}$ such that $\forall u \in res(r)$, $|r|(u\Sigma^{>k}) < \epsilon|r|(u\Sigma^*)$.*

Proof. For any integer k, let $f_k : CCH(r) \to \mathbb{R}$ be defined by $f_k(s) = |s|(\Sigma^{>k})$. For any $s, t \in CCH(r)$, we have $|f_k(s)-f_k(t)| \leq ||s-t||_1$. Hence, f_k is continuous for any k. Moreover, $lim_{k\to\infty} f_k(s) = 0$ for any $s \in CCH(r)$. Since $CCH(r)$ is compact, (f_k) converges uniformy to 0: for any $\epsilon > 0$, there exists $K \geq 0$ s.t. for any $k \geq K$ and any $s \in CCH(r)$, $|f_k(s)| < \epsilon$. Apply the result to $s = u^{-1}r$ and $\epsilon/|r|(\Sigma^*)$ to obtain the second result. □

Lemma 2. *Let $r \in \mathcal{A}^{rat}(\Sigma)$ and let $d = dim(LH(r))$. For all $\epsilon > 0$, there exists an integer N such that for any $u \in res(r)$, there exists $v_1, \ldots, v_d \in res(r)\cap\Sigma^{\leq N}$ and $\alpha_1, \ldots, \alpha_d > -\epsilon$ such that $\sum_{1\leq i\leq d}\alpha_i < 1+\epsilon$ and $u^{-1}r = \sum_{1\leq i\leq d}\alpha_i v_i^{-1}r$.*

Proof. Suppose that there exists $\epsilon > 0$ such that for any integer n, there exists $u_n \in res(r)$ such that for any $v_1, \ldots, v_d \in res(r) \cap \Sigma^{\leq n}$, $u_n^{-1}r = \sum_{i=1}^d \alpha_i v_i^{-1}r$ implies that there exists an index i s.t. $\alpha_i \leq -\epsilon$, or $\sum_{i=1}^d \alpha_i \geq 1+\epsilon$. Let n_k be a subsequence such that $u_{n_k}^{-1}r$ converges to an absolutely convergent series $s \in CCH(r)$.

Now, let $v_1, \ldots, v_{d-1}$ be such that $v_1^{-1}r, \ldots, v_{d-1}^{-1}r, s$ form a basis of $LH(r)$. For any integer k, let $u_{n_k}^{-1}r = \alpha_{1,k}v_1^{-1}r + \ldots + \alpha_{d-1,k}v_{d-1}^{-1}r + \alpha_{d,k}s$. Since $u_{n_k}^{-1}r$ converges to s, $\alpha_{i,k}$ converges to 0 for $i = 1, \ldots, d-1$ and $\alpha_{d,k}$ converges to 1 when k tends to infinity. Therefore, there should exist an integer K such that for any $k \geq K$, each coefficient of this combination is strictly greater that $-\epsilon$, and the sum of all its coefficient is strictly lower than $1+\epsilon$, which is contradictory. $\square$

3.2 A Particular Representation of Absolutely Convergent Rational Series

Let $A = \langle \Sigma, Q, \varphi, \iota, \tau \rangle$ be an MA. Let us denote by $|A|$ the MA defined by $|A| = \langle \Sigma, Q, |\varphi|, |\iota|, |\tau| \rangle$.

Lemma 3. $|r_A| \leq r_{|A|}$. *Hence, if* $r_{|A|}$ *is convergent, then* r_A *is absolutely convergent.*

Proof. Indeed,

$$|r_A(w)| = |\sum_{q,q' \in Q} \iota(q)\varphi(q,w,q')\tau(q')| \leq \sum_{q,q' \in Q} |\iota(q)\varphi(q,w,q')\tau(q')| = r_{|A|}(w).$$

$\square$

Lemma 4. *Let* $A = \langle \Sigma, Q, \varphi, \iota, \tau \rangle$ *be a MA. Suppose that* φ *takes only non negative values and that there exists an integer* k *such that for any state* q, $\varphi(q, \Sigma^k, Q) < 1$. *Then, the series* $r_{|A|}$ *is convergent.*

Proof. Let $R = \sup\{\varphi(q, \Sigma^h, Q) | q \in Q, h < k\}$ and $\rho = \sup\{\varphi(q, \Sigma^k, Q) | q \in Q\}$. From the hypothesis, $\rho < 1$.

Since for any state q and any integers $n > 0$ and $0 \leq h < k$, $\varphi(q, \Sigma^{nk+h}, Q) = \sum_{q' \in Q} \varphi(q, \Sigma^{(n-1)k+k}, q')\varphi(q', \Sigma^h, Q)$, it can easily be shown by induction on n that $\varphi(q, \Sigma^{nk+h}, Q) \leq R\rho^n$.

Let $I = Sup\{|\iota(q)| \text{ for } q \in Q\}$ and $T = Sup\{|\tau(q)| \text{ for } q \in Q\}$. We have

$$|r_A|(\Sigma^{nk+h}) \leq \sum_{q,q' \in Q} |\iota(q)|\varphi(q, \Sigma^{nk+h}, q')|\tau(q')| \leq IT \sum_{q \in Q} \varphi(q, \Sigma^{nk+h}, Q) \leq ITR|Q|\rho^n$$

$$\text{Therefore, } |r_A|(\Sigma^*) = \sum_{n \geq 0} \sum_{h=0}^{k-1} |r_A|(\Sigma^{nk+h}) \leq ITR|Q|k \sum_{h \geq 0} \rho^n = \frac{ITR|Q|k}{1-\rho}.$$

Note that if A is prefixial, there exists a unique state q such that $\iota(q) \neq 0$. So we can take $R = \sup\{\varphi(\varepsilon, \Sigma^h, Q) | q \in Q, h < k\}$ and we have $|r_A|(\Sigma^{nk+h}) \leq ITR\rho^n$. and $|r_A|(\Sigma^*) \leq \frac{ITRk}{1-\rho}$. $\square$

Theorem 1. *Let* $r \in \mathcal{A}^{rat}(\Sigma)$. *Let* $\rho < 1$. *There exists an integer* n, *and a prefixial MA* $A = \langle \Sigma, Q, \varphi, \iota, \tau \rangle$ *that computes* r *and such that* $\forall u \in Q$, $|\varphi|(u, \Sigma^n, Q) < \rho$. *Hence, the series* $r_{|A|}$ *is convergent.*

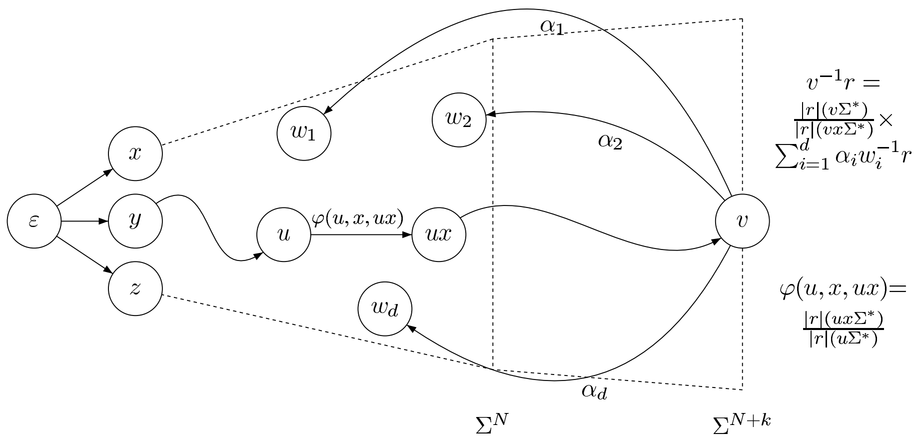

Fig. 1. Automaton built in the proof of Theorem 1

Proof. Let $r \in \mathcal{A}^{rat}(\Sigma)$ and let $d = dim(LH(r))$. Let $\rho < 1$ and let $\varepsilon = \rho/16$. From Lemma 1, let k be an integer such that $\forall u \in res(r)$, $\frac{|r|(u\Sigma^{\geq k})}{|r|(u\Sigma^*)} < \epsilon$. Let $\epsilon' = (d|\Sigma|)^{-1}$. From Lemma 2, let $N > k$ be an integer such that for all $u \in res(r)$, there exists $v_{u,1}, \ldots, v_{u,d} \in res(r) \cap \Sigma^{\leq N}$ and $\alpha_{u,1}, \ldots, \alpha_{u,d} > -\epsilon'$ such that $\sum_{i=1}^{d} \alpha_{u,i} < 1 + \epsilon' \leq 2$ and $u^{-1}r = \sum_{i=1}^{d} \alpha_{u,i} v_{u,i}^{-1} r$.

Let $Q = \Sigma^{N+k} \cap res(r)$. Let $f : Q \to \mathbb{R}$ be defined by $f(\varepsilon) = 1$ and $\forall ux \in Q, f(ux) = \frac{|r|(ux\Sigma^*)}{|r|(u\Sigma^*)}$ where $u \in \Sigma^*$ and $x \in \Sigma$. One can easily show that $\overline{f}(u) = |r|(u\Sigma^*)/|r|(\Sigma^*)$.

Let $F = \{ux | u \in Q, x \in \Sigma, ux \notin Q\}$. For any $ux \in F \cap res(r)$, there exist coefficients $(\alpha_{ux,v})_{v \in \Sigma^{\leq N}}$ such that (i) at most d coefficients $\alpha_{ux,v}$ are not null; (ii) $(ux)^{-1}r = \sum_{v \in \Sigma^{\leq N}} \alpha_{ux,v} v^{-1} r$; (iii) $\alpha_{ux,v} > -\epsilon'$; (iv) $\sum_{v \in \Sigma^{\leq N}} \alpha_{ux,v} < 1 + \epsilon'$. From $(ux)^{-1}r = \sum_{v \in \Sigma^{\leq N}} \alpha_{ux,v} v^{-1} r$, then $\dot{\overline{ux}} r = |r|(ux\Sigma^*) \sum_{v \in \Sigma^{\leq N}} \frac{\alpha_{ux,v}}{|r|(v\Sigma^*)} \dot{v} r$.

Let $g : Q \times F \to \mathbb{R}$ be defined by $g(ux, v) = \alpha_{ux,v} \frac{|r|(ux\Sigma^*)}{|r|(u\Sigma^*)}$ if $ux \in F \cap res(r)$ and 0 otherwise. One can check that the conditions on f and g stated in Section 2.2 are satisfied. (see Fig. 1)

From the proposition 1, the MA $A(\Sigma, Q, f, g, r) = \langle \Sigma, Q, \varphi, \iota, \tau \rangle$ computes r.

Let us list below some properties of A:

1. $\forall u \in Q \cap \Sigma^{<N+k}$ and $x \in \Sigma$, $\varphi(u, x, ux) = \frac{|r|(ux\Sigma^*)}{|r|(u\Sigma^*)} \leq 1$;
2. $\forall u \in Q \cap \Sigma^{<N+k}$ and $h < N + k - |u|$, $\varphi(u, \Sigma^h, Q) = \frac{|r|(u\Sigma^{\geq h})}{|r|(u\Sigma^*)} \leq 1$;
3. $\forall u \in Q \cap \Sigma^{\leq N}$, $\varphi(u, \Sigma^k, Q) = \frac{|r|(u\Sigma^{\geq k})}{|r|(u\Sigma^*)} < \varepsilon$;
4. $\forall u \in Q \cap \Sigma^{N+k}$ and $x \in \Sigma$, $\varphi(u, x, v) = \alpha_{ux,v} \frac{|r|(ux\Sigma^*)}{|r|(u\Sigma^*)} > -\varepsilon'$;
5. $\forall u \in Q \cap \Sigma^{N+k}$, $\varphi(u, \Sigma, Q) < 1 + \varepsilon' \leq 2$;
6. $\forall u \in \Sigma^{N+k}$, $(|\varphi|(u, \Sigma, Q) - \varphi(u, \Sigma, Q)) < 2d|\Sigma|\epsilon'$.

From these properties, one can deduce that:

a) For all $u \in \Sigma^h, h \leq N-k$ one have

$$|\varphi|(u, \Sigma^{2k}, Q) = \sum_{w \in \Sigma^k} \varphi(u, w, uw)\varphi(uw, \Sigma^k, Q) < \epsilon^2 < \rho \tag{1}$$

by applying twice the property 3.

b) For $u \in \Sigma^{N-k+1}$ one have

$$|\varphi|(u, \Sigma^{2k}, Q) < \epsilon(2 + 2d|\Sigma|\epsilon') < \rho. \tag{2}$$

Indeed, $|\varphi|(u, \Sigma^{2k}, Q) =$

$$\begin{aligned}
&\sum_{|w|=2k-1} \varphi(u, w, uw)[\varphi(uw, \Sigma, Q) + (|\varphi|(uw, \Sigma, Q) - \varphi(uw, \Sigma, Q))] \\
&\leq \sum_{|w|=2k-1} \varphi(u, w, uw)[2 + 2d|\Sigma|\epsilon'] \text{ from properties 5 and 6} \\
&\leq \varepsilon(2 + 2d|\Sigma|\epsilon') \text{ from properties 2 and 3.}
\end{aligned}$$

c) For $u \in \Sigma^h$ where $N-k+1 < h \leq N$ one have

$$|\varphi|(u, \Sigma^{2k}, Q) < \epsilon(2 + 2d|\Sigma|\epsilon') < \rho. \tag{3}$$

Indeed, $|\varphi|(u, \Sigma^{2k}, Q) =$

$$\begin{aligned}
&\sum_{|w|=N+k-h} \varphi(u, w, uw) \sum_{|v| \leq N} |\varphi|(uw, \Sigma, v)\varphi(v, \Sigma^{k+h-N-1}, Q) \\
&\leq \sum_{|w|=N+k-h} \varphi(u, w, uw) \sum_{|v| \leq N} |\varphi|(uw, \Sigma, v) \text{ from property 2} \\
&\leq \sum_{|w|=N+k-h} \varphi(u, w, uw)[\varphi(uw, \Sigma, Q) + (|\varphi|(uw, \Sigma, Q) - \varphi(uw, \Sigma, Q))] \\
&\leq \sum_{|w|=N+k-h} \varphi(u, w, uw)[2 + 2d|\Sigma|\epsilon'] \text{ from properties 5 and 6} \\
&\leq \varepsilon(2 + 2d|\Sigma|\epsilon') \text{ from properties 2 and 3.}
\end{aligned}$$

d) For $u \in \Sigma^h, N < h \leq N+k$ one have

$$|\varphi|(u, \Sigma^{2k}, Q) < \varepsilon(2 + 2d|\Sigma|\epsilon')^2 < \rho. \tag{4}$$

Indeed,

$$|\varphi|(u, \Sigma^{2k}, Q) < \sum_{w \in \Sigma^{N+k-h}} \varphi(u, w, uw) \sum_{v \in \Sigma^{\leq N}} |\varphi|(uw, \Sigma, v)|\varphi|(v, \Sigma^{h+k-N-1}, Q).$$

If $|v| \leq 2N - h + 1$, $|v| + h + k - N - 1 \leq N + k$, and from Property 3,

$$|\varphi|(v, \Sigma^{h+k-N-1}, Q) \leq \epsilon$$

If $|v| > 2N - h + 1$, $|\varphi|(v, \Sigma^{h+k-N-1}, Q)$

$$\begin{aligned}
&\leq \sum_{|w|=N+k-|v|} \varphi(v, w, vw) \sum_{v' \in \Sigma^{\leq N}} |\varphi|(vw, \Sigma, v')\varphi(v', \Sigma^{|v|+h-2N-2}, Q) \\
&\leq \sum_{|w|=N+k-|v|} \varphi(v, w, vw)|\varphi|(vw, \Sigma, Q) \\
&\leq (2 + 2d|\Sigma|\epsilon')\varphi(v, \Sigma^{N+k-|v|}, Q) \\
&\leq \varepsilon(2 + 2d|\Sigma|\epsilon').
\end{aligned}$$

That is, in all cases, $|\varphi|(v, \Sigma^{h+k-N-1}, Q) \leq \varepsilon(2 + 2d|\Sigma|\epsilon')$ and

$$\begin{aligned}
|\varphi|(u, \Sigma^{2k}, Q) &< \varepsilon(2 + 2d|\Sigma|\epsilon') \sum_{w \in \Sigma^{N+k-h}} \varphi(u, w, uw) \sum_{v \in \Sigma^{\leq N}} |\varphi|(uw, \Sigma, v) \\
&\leq \varepsilon(2 + 2d|\Sigma|\epsilon')^2
\end{aligned}$$

We have proved that for any $u \in Q$, $|\varphi|(u, \Sigma^{2k}, Q) < \rho$. Hence, from lemma 4, the series $r_{|A|}$ is convergent. □

Spectral radius of a matrix, joint or generalized spectral radius of a set of matrices are tools used to study asymptotic properties of powers or products of matrices. A good introduction on spectral radii can be found in the first chapters of [8].

Definition 2. *Let* $r \in \mathbb{R}^{rat}\langle\langle\Sigma\rangle\rangle$. *We define the* absolute spectral radius

$$\rho_{|r|} = \limsup_n \left(|r|(\Sigma^n)\right)^{1/n}$$

Proposition 2. *Let* $r \in \mathbb{R}^{rat}\langle\langle\Sigma\rangle\rangle$. $r \in \mathcal{A}^{rat}(\Sigma)$ *if and only if* $\rho_{|r|} < 1$.

Proof. Let $r \in \mathbb{R}^{rat}\langle\langle\Sigma\rangle\rangle$. Suppose that $\rho_{|r|} < 1$, then there exists ρ s.t. $\rho_{|r|} < \rho < 1$ and $n \in \mathbb{N}$ such that $|r|(\Sigma^{\geq n}) < \sum_{i=n}^{\infty} \rho^i$. Thus r is absolutely convergent. Suppose now that $r \in \mathcal{A}^{rat}(\Sigma)$. By the theorem 1, there exists a prefixial automaton $A = \langle \Sigma, Q, \varphi, \iota, \tau \rangle$ that computes r, $\rho < 1$ and an integer n such that for every state q, $|\varphi_A|(q, \Sigma^n, Q) < \rho$. Hence, from Lemma 4, $|r|(\Sigma^m) = O(\rho^{m/n})$ and therefore, $\rho_{|r_A|} \leq \rho^{1/n} < 1$. □

4 Decidability

We prove in this section that there exists an algorithm that takes a multiplicity automaton A as input and halts iff r_A is absolutely convergent. In other words, the class $\mathcal{A}(\Sigma)$ is semi-decidable.

Theorem 2. *The class $\mathcal{A}^{rat}(\Sigma)$ is semi-decidable.*

Proof. Let r be a rational series and let A be a MA that computes r. Queries such as: What is the dimension d of $LH(r)$? Does the word u belongs to $res(r)$? Given $u_1, \ldots, u_d \in res(r)$, is $\{\dot{u_1}r, \ldots, \dot{u_d}r\}$ a basis of $LH(r)$? can be answered by using A.

We consider the countable class $\mathcal{S}_r$ composed of all the prefixial NFA $\langle \Sigma, Q, \delta, I, T\rangle$ that satisfy the following properties:

- there exists two integers N and k such that $Q = \Sigma^{\leq N+k} \cap res(u)$;
- for any $u \in Q$ and $x \in \Sigma$ such that $ux \in res(u) \backslash Q$, the set $\delta(u,x)$ is included in $\Sigma^{\leq N}$ and the set $\{\dot{v}r | v \in \delta(u,x)\}$ forms a basis of $LH(r)$. In particular, $\delta(u,x)$ contains exactly d elements.

Let $(A_m)_{m\in\mathbb{N}}$ be an enumeration of $\mathcal{S}_r$.

Now let $(f_n)_{n\in\mathbb{N}}$ be a family of functions defined as follows: for any integer n, $f_n : res(r) \to \mathbb{R}$, $f_n(\varepsilon) = 1$ and for any $x \in \Sigma$ and $ux \in res(r)$, $f_n(ux) = \frac{|r|(ux\Sigma^{\leq n})}{|r|(u\Sigma^{\leq n+1})}$ if $|r|(u\Sigma^{\leq n}) \neq 0$ and $f_n(ux) = 1$ otherwise. Note that $\lim_{n\to\infty} f_n(ux) = \frac{|r|(ux\Sigma^*)}{|r|(u\Sigma^*)}$.

Let $m \in \mathbb{N}$. For any integer n, we define a MA $A_{m,n}$ whose support is equal to $A_m = \langle \Sigma, Q_m, \delta_m, \{\varepsilon\}, T_m\rangle$. Let $f : Q_m \to \mathbb{R}$ be defined by $f(u) = f_n(u)$. Let $F_m = \{ux | u \in Q_m, x \in \Sigma, ux \notin Q_m\}$ and let $g : F_m \times Q_m \to \mathbb{R}$ be defined by $g(ux,v) = 0$ if $v \notin \delta_m(u,x)$ and $\dot{\overline{ux}}r = \sum_{v\in Q} g(ux,v)\frac{\overline{f(u)}}{\overline{f(v)}}\dot{v}r$: note that g is completely determined by A_m, f_n and r since for any $ux \in F_m$, $\{\dot{v}r | v \in \delta(u,x)\}$ forms a basis of $LH(r)$. We let $A_{m,n} = A(\Sigma, Q_m, f, g, r)$.

Now, consider the following algorithm:

- enumerate $(m,n,k) \in \mathbb{N}$
- for each tuple (m,n,k), build the MA $A_{m,n} = \langle \Sigma, Q_m, \varphi_{m,n}, \iota_{m,n}, \tau_{m,n}\rangle$
- if $Max_{q\in Q_m}\{|\varphi_{m,n}|(q, \Sigma^k, Q_m)\} < 1$, halts.

If r is absolutely convergent, from Theorem 1, there exists a prefixial MA $A = \langle \Sigma, Q_m, \varphi, \iota, \tau\rangle$ that computes r, whose support is A_m for some integer m and such that $Max_{q\in Q_m}\{\varphi(q, \Sigma^k, Q_m)\} < \rho$ for some integer k and some $\rho < 1$. Since $Max_{q\in Q_m}\{\varphi(q, \Sigma^k, Q_m)\}$ is a continuous function in the parameters of A, and since $\lim_{n\to\infty} \varphi_{m,n}(q,x,q') = \varphi(q,x,q')$ for any states q, q' and any letter x, for any $\rho < \rho' < 1$, there exists an integer N such that $Max_{q\in Q_m}\{\varphi_{m,n}(q, \Sigma^k, Q_m)\} < \rho'$ for any integer $n \geq N$. Hence, the algorithm halts on input r.

Clearly, from Lemma 3, since any MA $A_{m,n}$ computes r, the algorithm does not halt if r is not absolutely convergent. □

5 Approximation and L_1-Distance

To our knowledge, the sum $|r|(\Sigma^*)$ cannot be exactly computed. Nevertheless, it is possible to estimate it by a lower bound since for any integer n, $|r|(\Sigma^{\leq n}) \leq$

$|r|(\Sigma^*)$ and $\lim_{n\to\infty} |r|(\Sigma^{\leq n}) = |r|(\Sigma^*)$. We will now define upper bounds of $|r|(\Sigma^*)$ which tend to $|r|(\Sigma^*)$. Hence, it is possible to bound the error made by the approximation to any accuracy rate.

We first give a variation of Theorem 1.

Lemma 5. *Let $r \in \mathcal{A}^{rat}(\Sigma)$. For any $u \in res(r)$ and any $x \in \Sigma$, let $f_n(ux) = \frac{|r|(ux\Sigma^{\leq n})}{|r|(u\Sigma^{\leq n+1})}$ if $|r|(u\Sigma^{\leq n+1}) \neq 0$ and $f_n(ux) = 1$ otherwise: $\lim_{n\to\infty} f_n(ux) = f(ux) = \frac{|r|(ux\Sigma^*)}{|r|(u\Sigma^*)}$. Let $T > |r|(\Sigma^*)$.*

For any integers N and k, let $D : \Sigma^{N+k+1} \to 2^{\Sigma^{\leq N}}$ be such that for any $ux \in \Sigma^{N+k+1}$, $\{\dot{v}r | v \in D(ux)\}$ forms a basis of $LH(r)$. In particular, $|D(ux)| = d = dim(LH(r))$. Given N, k, D and f_n, there exists a unique function $g : \Sigma^{N+k+1} \times \Sigma^{\leq N} \to \mathbb{R}$ such that $g(ux, v) = 0$ if $v \notin D(ux)$ and such that $A(\Sigma, Q, f_n, g, r)$ computes r (see Proposition 1). Let us denote it by $A(N, k, D, f_n)$.

Now given $\rho < 1$, there exists k_0, N_0 such that $\forall N > N_0, \forall k > k_0$ there exists n and D such that $A = A(N, k, D, f_n)$ satisfies:

- $\varphi_{|A|}(u, \Sigma^{2k}, Q) < \rho$
- $\forall u \in \Sigma^{N+k}, |\tau(u)| < T$

Proof. By considering the proof of Theorem 1, it can be proved that there exists integers N_0 and k_0 such that $\forall N > N_0, \forall k > k_0$ there exists $D : \Sigma^{N+k+1} \to 2^{\Sigma^{\leq N}}$ such that $A = A(N, k, D, f)$ satisfies $\varphi_{|A|}(u, \Sigma^{2k}, Q) < \rho/2$ and $\forall u \in \Sigma^{N+k}, |\tau(u)| \leq |r|(\Sigma^*)$. Since $\varphi_{|A(N,k,D,f_n)|}(u, \Sigma^{2k}, Q)$ and $|\tau_{|A(N,k,D,f_n)|}(u)|$ are continuous relatively to the transition coefficients, there exists an integer n such that the conclusion holds. □

Proposition 3. *Let $r \in \mathcal{A}^{rat}(\Sigma)$. Let $\mathbb{A} = \{A(N, k, D, f_n)/N, k, n \in \mathbb{N}, D \in \mathcal{P}(\Sigma^N)\}$. Then $\inf(r_{|A|}(\Sigma^*))_{A\in\mathbb{A}} = |r|(\Sigma^*)$.*

Proof. Let N_0 and k_0 be such as in Lemma 5. We define a sequence $(A_z)_{z\in\mathbb{N}}$ of automata:

- $N_z = N_0 + z$, $k_z = k_0$
- $\forall z$, n_z and D_z are such as in Lemma 5, $A_z = A(N_z, k_z, D_z, f_{n_z})$.

As N_z grows, $r_{|A_z|}$ converges pointwisely to $|r|$. We have $\|\tau_z\|_\infty < T$ by construction. We have $R = sup_{h<2k_z}(|\varphi_z|(\varepsilon, \Sigma^{2k_z}, Q)) < 1$, as $|\varphi_z|(u, \Sigma, Q)) < 1$ for $u \in \Sigma^{<N_z+k_z}$. Applying Lemma 4, we have $r_{|A_z|}(\Sigma^{>n}) < T\rho^{n/2k_0-1}$ and therefore $|r|(\Sigma^{>n}) < T\rho^{n/2k_0-1}$. Now, $|r_{|A_z|}(\Sigma^*) - |r|(\Sigma^*)| \leq \sum_{w\in\Sigma^*} |r_{|A_z|}(w) - |r|(w)| \leq \sum_{w\in\Sigma^{\leq Z_1}} |r_{|A_z|}(w) - |r|(w)| + \sum_{w\in\Sigma^{>Z_1}} r_{|A_z|}(w) + \sum_{w\in\Sigma^{>Z_1}} |r|(w)$. For $\epsilon > 0$, one can find Z_1 such that both $\sum_{w\in\Sigma^{>Z_1}} r_{|A_z|}(w) < \epsilon/3$ and $\sum_{w\in\Sigma^{>Z_1}} |r|(w) < \epsilon/3$. Finally, as $r_{|A_z|}$ converges pointwisely to $|r|(w)$, one can find Z_2 such that $\sum_{w\in\Sigma^{\leq Z_1}} |r_{|A_z|}(w) - |r|(w)| < \epsilon/3$, and we can conclude. □

Proposition 4. *Let A_Z be an enumeration of the set $\{A(N, k, D, n)/N, k, n \in \mathbb{N}, D \in \mathcal{P}(\Sigma^N)\}$. Let $G_0 = r_{|A(N_0,k_0,D_0,f_{n_0})|}(\Sigma^*)$. Let $G_z = r_{|A_z|}(\Sigma^*)$ if it exists, and if $G_z < G_{z-1}$, $G_z = G_{z-1}$ otherwise. Then $\lim_{z\to\infty} G_z = |r|(\Sigma^*)$.*

Proof. Straightforward from Proposition 3. □

Theorem 3. *Let $r \in \mathcal{A}^{rat}(\Sigma)$. For any $\varepsilon > 0$, it is possible to compute an estimate $\widehat{|r|(\Sigma^*)}$ of $|r|(\Sigma^*)$ such that $\left|\widehat{|r|(\Sigma^*)} - |r|(\Sigma^*)\right| < \varepsilon$. If the L_1-distance d of two rational series r and s is finite, for any $\varepsilon > 0$, it is possible to compute an estimate $\hat{d}$ of d such that $|\hat{d} - d| < \varepsilon$.*

Proof. Using Proposition 4, find G_z and n such that $G_z - |r|(\Sigma^{\leq n}) < 2\varepsilon$. For the second part, apply the result to $r - s$. □

6 Conclusion

In this paper, we have proved that it is semi-decidable whether a rational series is absolutely convergent. Then, given an absolutely convergent rational series r, we have provided an algorithmic way to estimate the sum $|r|(\Sigma^*)$ to any accuracy rate. We do not know whether $\mathcal{A}^{rat}(\Sigma)$ is decidable. We conjecture that it should be decided that a rational series r is not absolutely convergent when $\rho_{|r|} > 1$ but the case $\rho_{|r|} = 1$ is likely to be difficult. We intend to study the links between the spectral radius we have defined and the joint or generalized spectral radii. Finally, we are currently looking for more efficient algorithms and heuristics than those we have described to approximate the L1 distance of two rational series, even if there is no hope to find efficient algorithms in the worst case.

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